18.06 Final Exam Professor Strang May 18, 2011

Your PRINTED name is ______________________________ **1.**

Your Recitation Instructor (and time) is ______________________ **2.**

Instructors: (Hezari)(Pires)(Sheridan)(Yoo) **3.**

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Please show enough work so we can see your method and give due credit.

1. (a) For this matrix A, find the usual P (permutation) and L and U so that $PA = LU$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 2 \\ 3 & 4 & 7 & 3 \end{bmatrix}.$$

(b) Find a basis for the nullspace of A.

(c) The vector (b_1, b_2, b_3) is in the column space of A provided it is orthogonal to __________ (give a numerical answer).

2. (a) Compute the 4 by 4 matrix P that projects every vector in R^4 onto the column space of A:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(b) What are the four eigenvalues of P? Explain your reasoning.

(c) Find a unit vector u (length 1) that is as far away as possible from the column space of A.

3. Suppose A is an m by n matrix and its pivot columns (not free columns) are $c_1, c_2, \ldots, c_r$. Put these columns into a matrix C.

   (a) Every column of A is a ____________ of the columns of C. *How would you produce from this a matrix R so that $A = CR$?* Explain how to construct R.

   (b) Using C from part (a) factor the following matrix A into CR, where C has independent columns and R has independent rows.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

4. (a) Find the cofactor matrix C for this matrix A. (The i, j entry of C is the cofactor including $\pm$ sign of the i, j position in A.)

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) If a square matrix B is invertible, how do you know that its cofactor matrix is invertible?

(c) True or false **with a reason**, if B is invertible with cofactor matrix C:

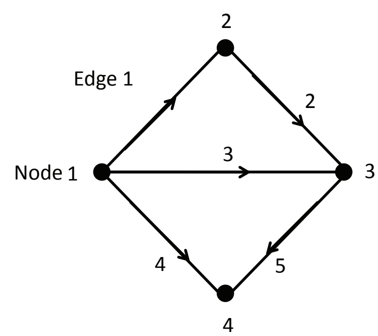
$$\text{determinant of} \quad B^{-1} = \frac{\text{determinant of} \quad C}{\text{determinant of} \quad B}$$

5. (a) Find the eigenvalues λ_1, λ_2, λ_3 and a full set of independent eigenvectors x_1, x_2, x_3 (if possible) for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(b) Suppose $u_0 = 3x_1 + 7x_2 + 5x_3$ is a combination of your eigenvectors of A: Find $A^k u_0$. If $\|v\|$ is the length of v, find the limit of $\frac{\|A^{k+1}u_0\|}{\|A^k u_0\|}$ as $k \to \infty$.

6. (a) For this directed graph, write down the 5 by 4 incidence matrix A. **Describe the nullspace of A.**



(b) Find the matrix $G = A^T A$. Is this matrix G positive definite? (Explain why or why not.) The first entry is $G_{11} = 3$ because the graph has ____________.

(c) What is the sum of the squares of the singular values of A? *Hint:* Remember that those numbers σ^2 are ____________.

7. Suppose A is a positive definite symmetric matrix with n different eigenvalues: $Ax_i = \lambda_i x_i$.

(a) What are the properties of those λ's and x's? How would you find an orthogonal matrix Q so that $A = Q\Lambda Q^T$ with $\Lambda =$ diag $(\lambda_1, \ldots, \lambda_n)$?

(b) I am looking for a symmetric positive definite matrix B with $B^2 = A$ (a square root of A). What will be the eigenvectors and eigenvalues of B? Can you find a formula for B using Q and Λ from part (a)?

(c) What are the eigenvalues and eigenvectors of the matrix e^{-A}? Is this matrix also positive definite and why?

8. Suppose the 2 by 3 matrix A has $Av_1 = 3u_1$ and $Av_2 = 5u_2$ with orthonormal v_1, v_2 in R^3 and orthonormal u_1, u_2 in R^2.

(a) Describe the nullspace of A.

(b) Find the eigenvalues of $A^T A$.

(c) Find the eigenvalues **and eigenvectors** of AA^T.

9. (a) The **index** of a matrix A is the dimension of its nullspace minus the dimension of the nullspace of A^T. If A is a 9 by 7 matrix of rank r, what is its index?

(b) Suppose M is the vector space consisting of all 2 by 2 matrices. (So those matrices are the "vectors" in M.) Write down a basis for this vector space M: linearly independent and spanning the space M.

(c) $S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is a specific matrix in M. For every 2 by 2 matrix A, the transformation T produces $T(A) = S^{-1}AS$. Is this a **linear** transformation? What tests do you have to check?